Semantics: Application to C Programs

Lecture

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SoSy-Lab Software Systems

Organization

Lecture and Tutorial

Lecture
Feb 27, 2019, 10:00 – 14:00
Munich, Oettingenstr. 67, C003

Tutorial
Feb 27, 2019, 14:00 – 15:30
Munich, Oettingenstr. 67, C003

Course Material

https://www.sosy-lab.org/Teaching/2019-SS-Semantik/

Introduction

Software Analysis

Computes an (over-)approximation of a program's **behavior**.

Applications

- Optimization
- Correctness
  (i.e. whether program satisfies a given property)

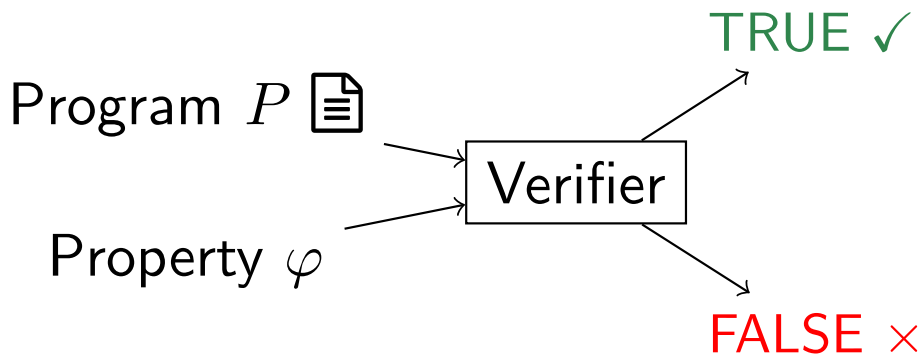
Software Verification

Formally proves whether a program P satisfies a property φ.

- Requires program semantics, i.e., meaning of program
- Relies on mathematical methods,
  - logic
  - induction
  - ...

Software Verification

Formally proves whether a program P satisfies a property φ.



Disprove (×) Find a program execution (**counterexample**) that violates the property φ

Prove (✓) Show that **every** execution of the program satisfies the property φ.

What Could an Analysis Find out?

```
double divTwiceCons(double y) {
    int cons = 5;
    int d = 2*cons;
    if (cons != 0)
        return y/(2*cons);
    else
        return 0;
}
```

Some Analysis Results

```
double divTwiceCons(double y) {
    int cons = 5;
    // expression 2*cons has value 10
    // variable d not used
    int d = 2*cons;
    if (cons != 0)
        // expression 2*cons evaluated before
        return y/(2*cons);
    else
        // dead code
        return 0;
}
```

One Resulting Code Optimization

```
double divTwiceCons(double y) {
    int cons = 5;
    // expression 2*cons has value 10
    // variable d not used
    int d = 2*cons;
    if (cons != 0)
        // expression 2*cons evaluated before
        return y/(2*cons);
    else
        // dead code
        return 0;
}
```

```
double divTwiceConsOptimized(double y) {
        return y/10;
}
```

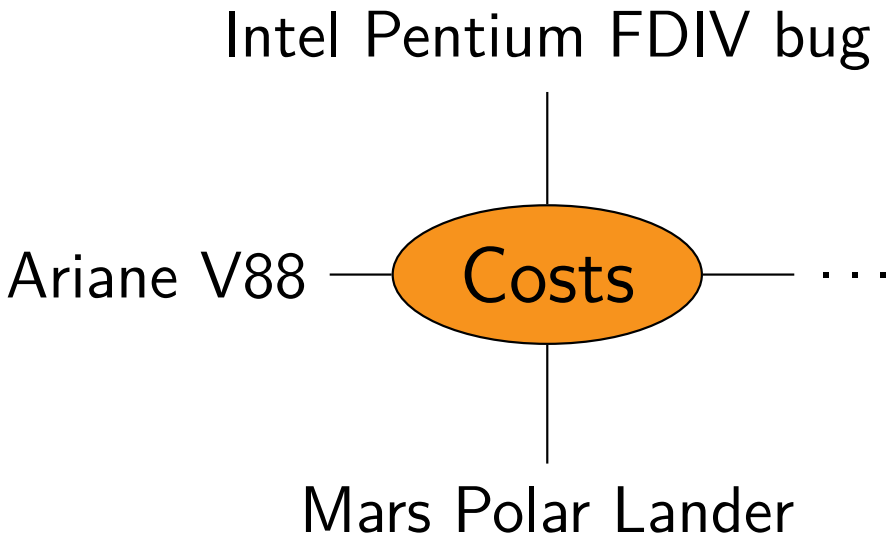
Does This Code Work?

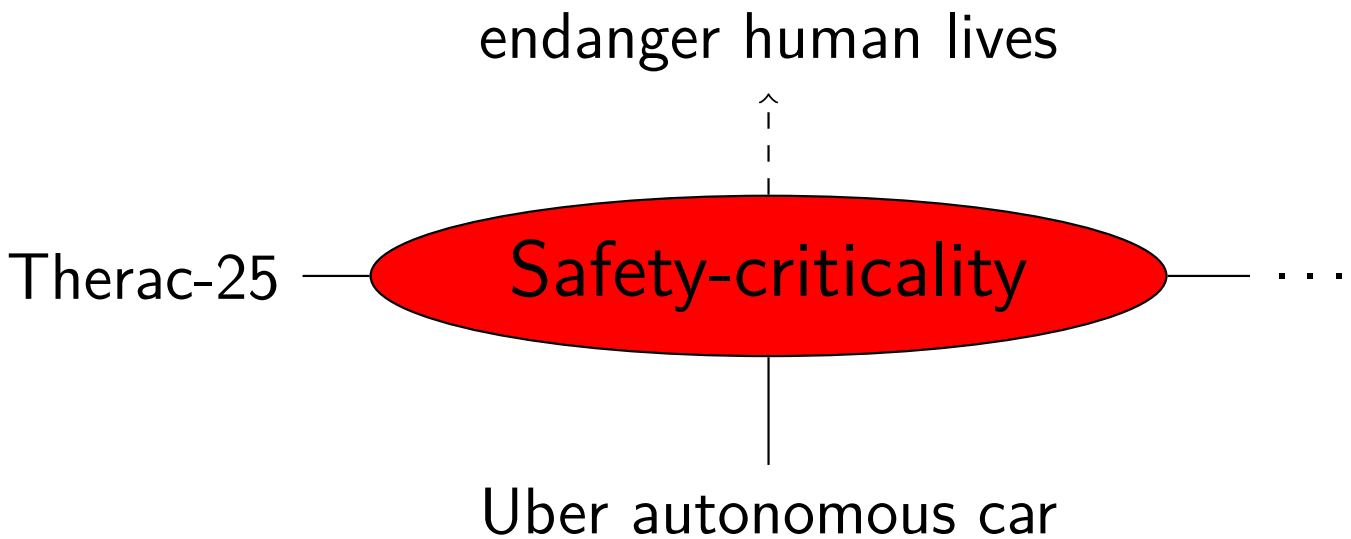
```
double avgUpTo(int[] numbers, int length) {
    double sum = 0;
    for(int  i=0;i<length;i++)
        sum += numbers[i];
    return sum/(double)length;
}
```

Problems With This Code

```
double avgUpTo(int[] numbers, int length) {
    double sum = 0;
    for(int  i=0;i<length;i++)
        // possible null pointer access (numbers==null)
        // index out of bounds (length>numbers.length)
        // integer overflow
        sum += numbers[i];
    // division by zero (length==0)
    return sum/(double) length;
}
```

Why Should One Care for Bugs?





Analysis and Verification Tools

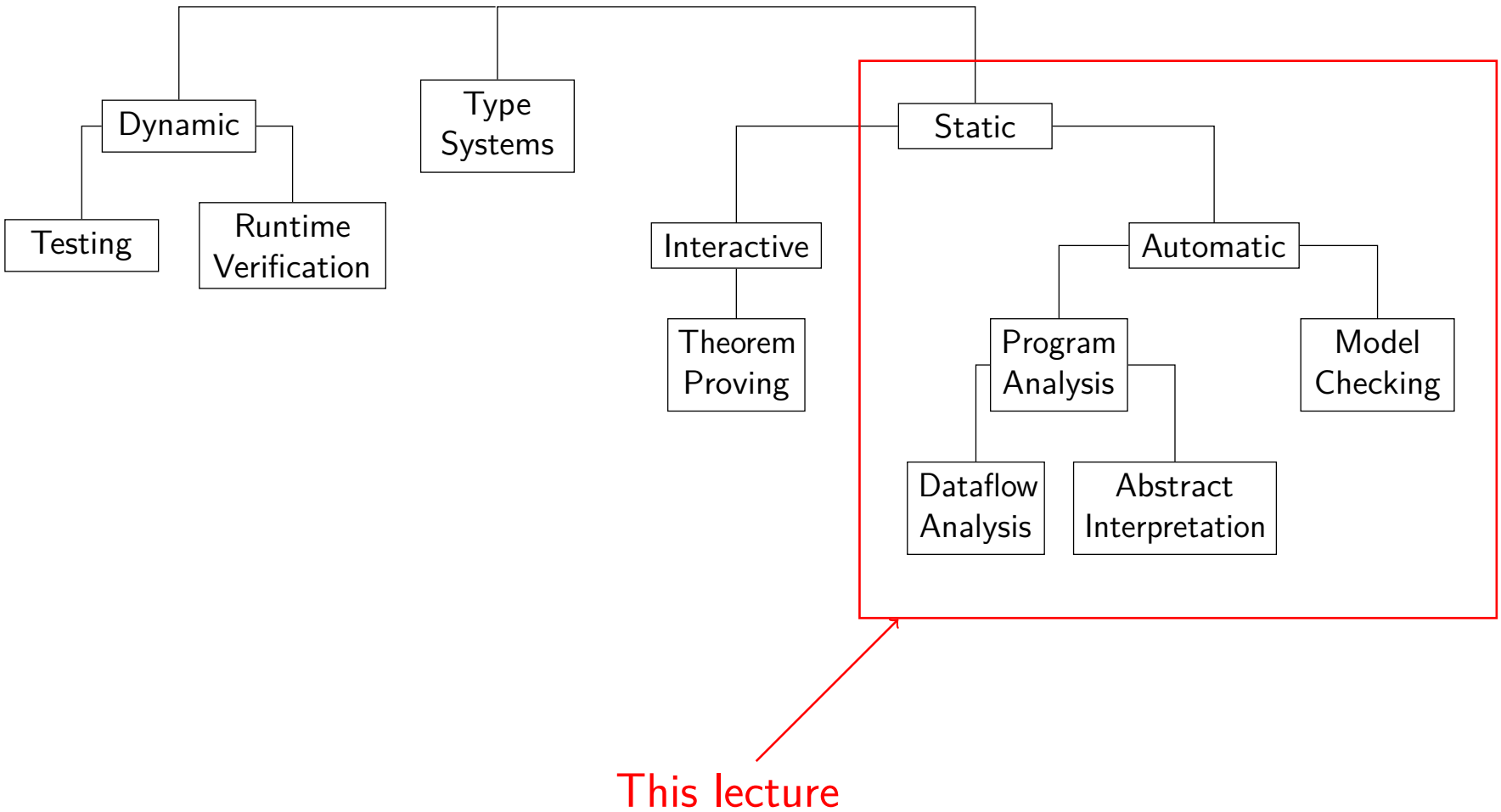
Sapienz Klee PeX

Infer Lint Error Prone SLAM

CBMC SpotBugs UltimateAutomizer

CPAchecker ...

Overview on Analysis and Verification Techniques



Why Different Static, Automatic Techniques?

Theorem of Rice

Any non-trivial, semantic property of programs is undecidable.

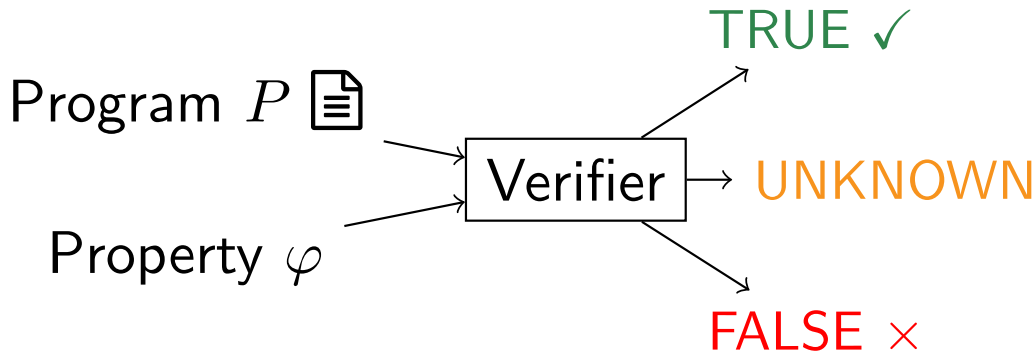
Consequences

Techniques are

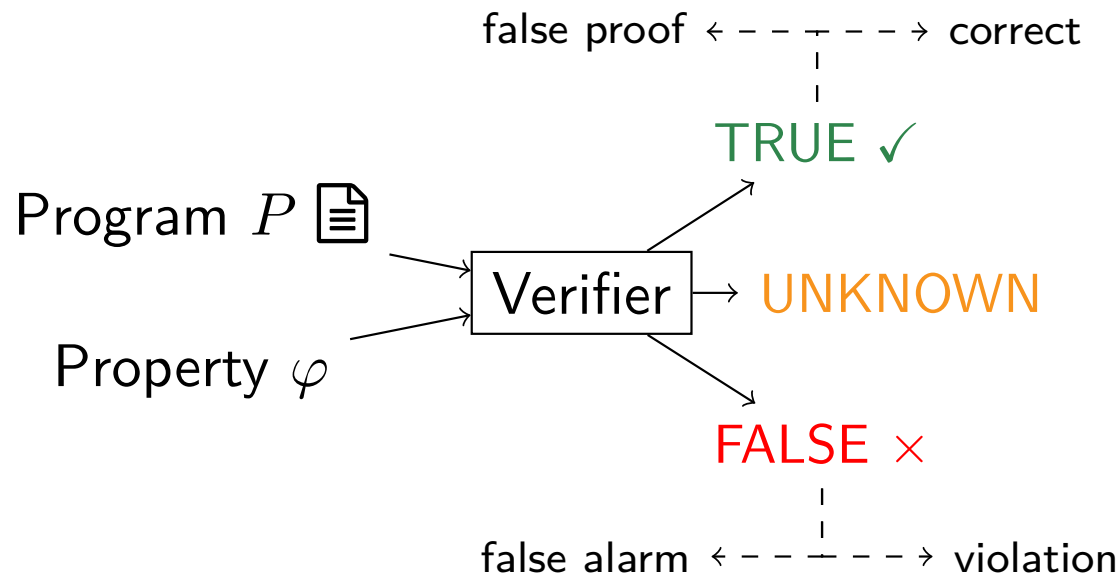
- incomplete, e.g. answer UNKNOWN, or
- unsound, i.e., report
  - false alarms (non-existing bugs),
  - false proofs (miss bugs).

Verifier Design Space

Ideal verifier

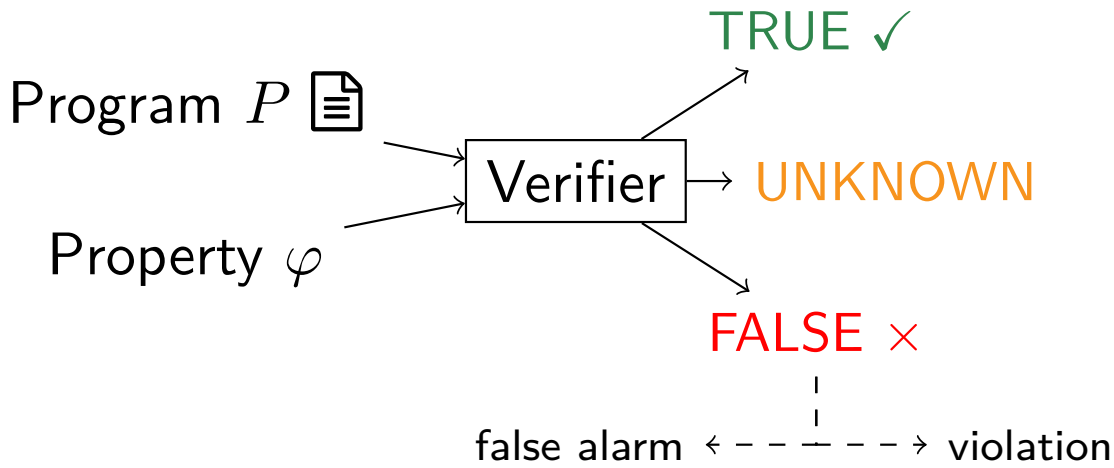


Unreliable verifier

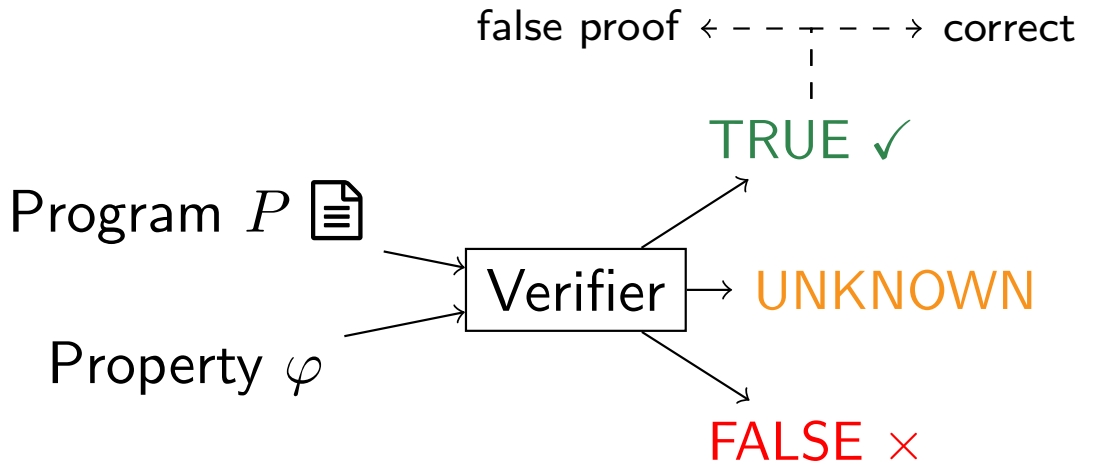


Verifier Design Space

- Overapproximating verifier (superset of program behavior)
  without precise counterexample check



- Underapproximating verifier (subset of program behavior)



Other Reasons to Use Different Static Techniques

- State space grows exponentially with number of variables
- (Syntactic) paths grow exponentially with number of branches

$\Rightarrow$ Precise techniques may require too many resources (memory, time,...)

$\Rightarrow$ Trade-off between precision and costs

Flow-Insensitivity

Order of statements not considered

E.g., does not distinguish between these two programs

```
x=0;          x=0;
y=x;          x=x+1;
x=x+1;        y=x;
```

$\Rightarrow$ very imprecise

Flow-Sensitivity Plus Path-Insensitivity

- Takes order of statements into account
- Mostly, ignores infeasibility of syntactical paths
- Ignores branch correlations

E.g., does not distinguish between these two programs

```
if (x>0)
    y=1;
else
    y=0;
if (x>0)
    y=y+1;
else
    y=y+2;
```

```
if (x>0)
    y=1;
else
    y=0;
if (x>0)
    y=y+2;
else
    y=y+1;
```

Path-Sensitivity

- Takes (execution) paths into account
- Excludes infeasible, syntactic paths (not necessarily all infeasible ones)
- Covers flow-sensitivity

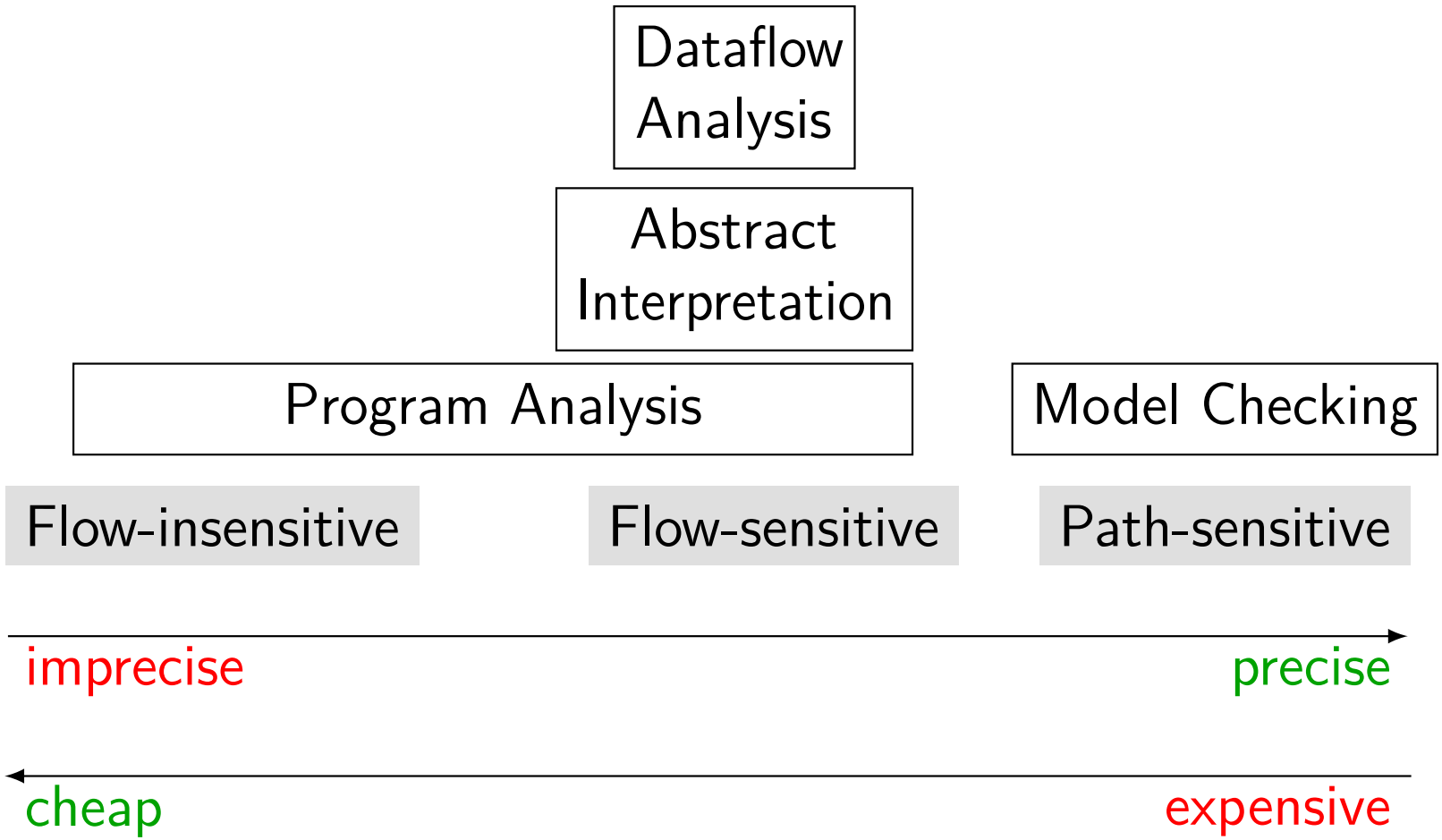
```
if (x>0)
    y=1;
else
    y=0;
if (x>0)
    y=y+2;
else
    y=y+1;
```

To detect that y has value 0, 1, or 3

- must exclude infeasible, syntactic path along first else-branch and second if-branch
- need to detect correlation between the if-conditions
- requires path-sensitivity

$\Rightarrow$ very precise

Precision vs. Costs



Program Syntax and Semantics

Programs

Theory: simple while-programs

- Restriction to **integer** constants and variables
- Minimal set of statements (assignment, if, while)
- Techniques easier to teach/understand

Practice: C programs

- Widely-used language
- Tool support

While-Programs

- Arithmetic expressions
  aexpr := $\mathbb{Z}$ | `var` | -aexpr | aexpr op_a aexpr
  op_a standard arithmetic operation like $+, -, /, \%, \ldots$
- Boolean expressions
  bexpr := aexpr | aexpr op_c aexpr | !bexpr | bexpr op_b bexpr
  - integer value $0 \equiv$false, remaining values represent true
  - op_c comparison operator like $<, <=, >=, >, ==, !=$
  - op_b logic connective like $\&\&(\wedge), ||\ (\vee)$, ^ (xor), $\ldots$
- Program
  S:= `var=aexpr;` | `while` bexpr S | `if` bexpr S `else` S |
  `if` bexpr S | S;S

Syntax vs. Semantics

Syntax

Representation of a program

Semantics

Meaning of a program

How to Represent a Program?

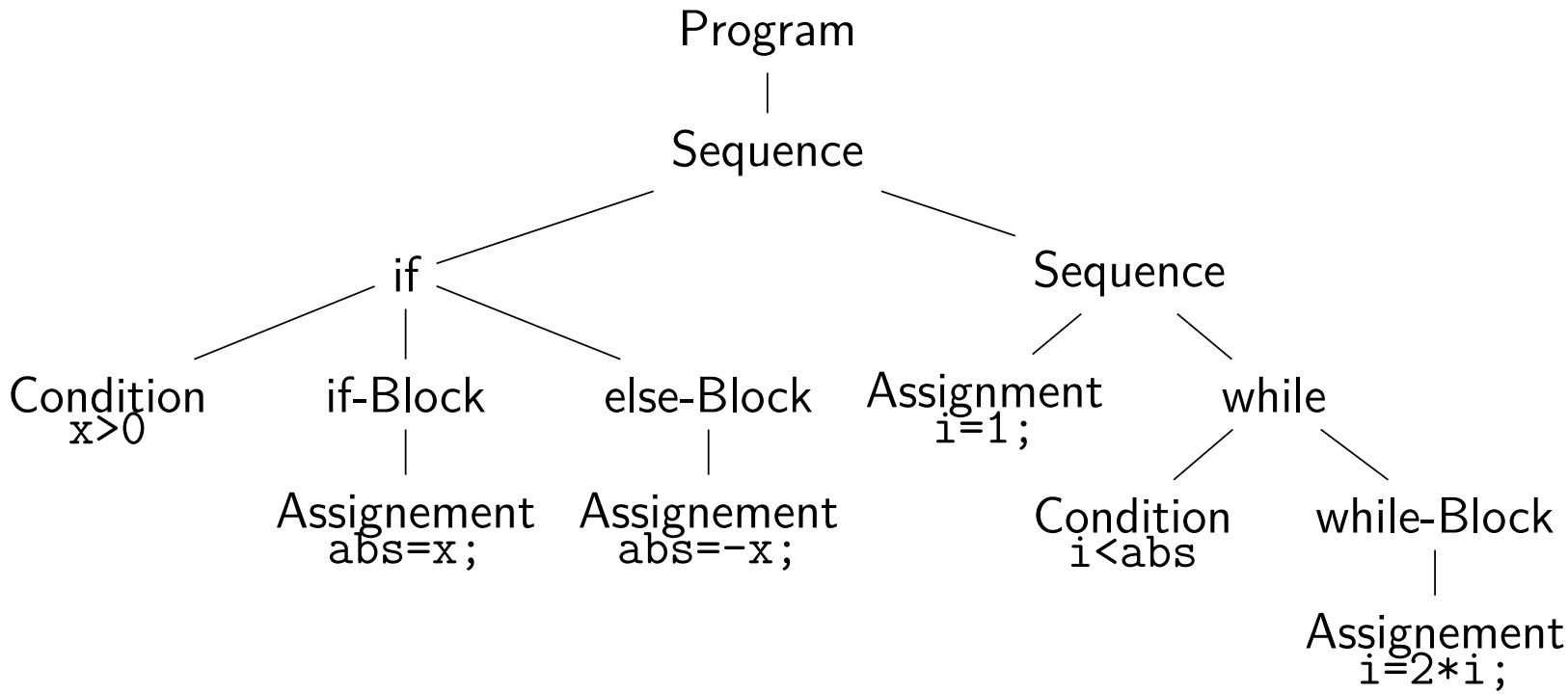
1. Source code

```
if (x>0)
    abs = x;
else
    abs = -x;
i = 1;
while(i<abs)
    i = 2*i;
```

- Basically sequence of characters
- No explicit information about the structure or paths of programs

How to Represent a Program?

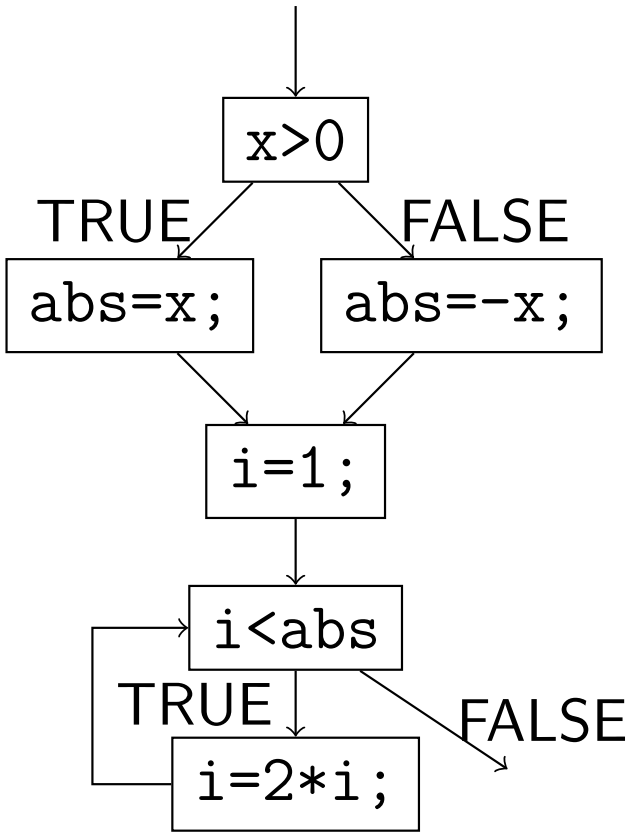
2. Abstract-syntax tree (AST)



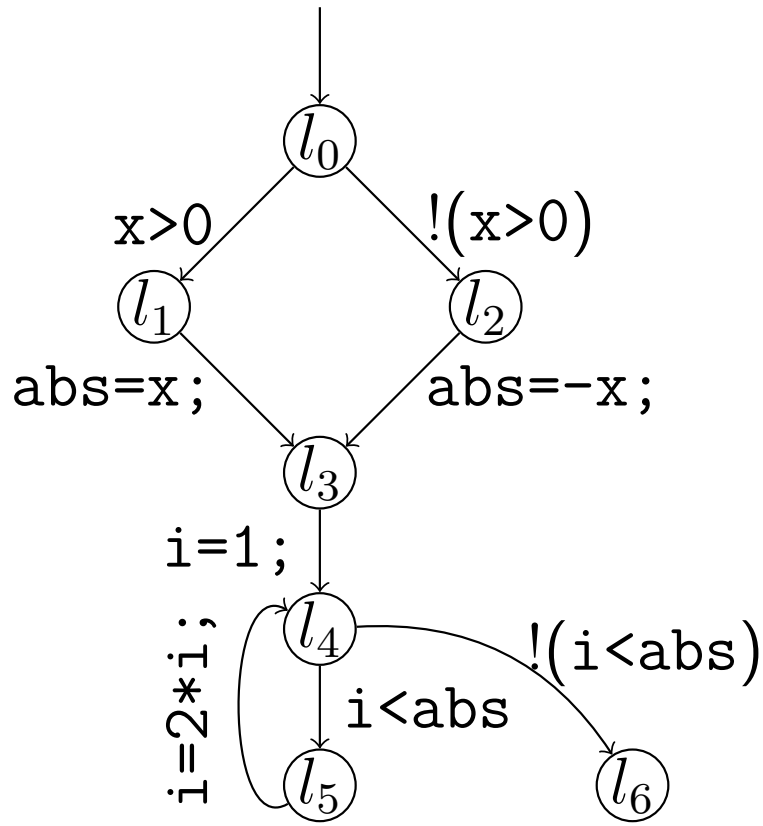
- Hierarchical representation
- Flow, paths hard to detect

How to Represent a Program?

3. Control-flow graph



4. Control-flow automaton



Control-Flow Automaton

Definition

A *control-flow automaton* (CFA) is a three-tuple $P = (L, l_0, G)$ consisting of

- the set L of program locations (domain of program counter)
- the initial program location $l_0 \in L$, and
- the control-flow edges $G \subseteq L \times Ops \times L$.

Operations Ops

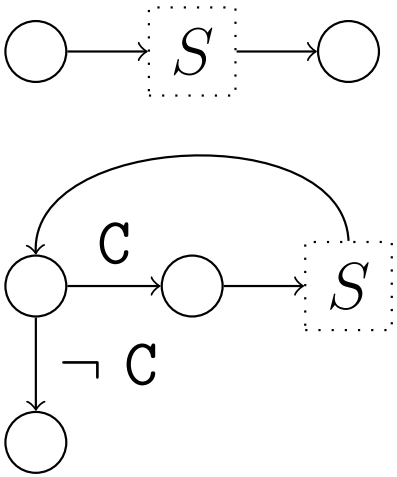
Two types

- Assumes (boolean expressions)
- Assignments (`var=aexpr;`)

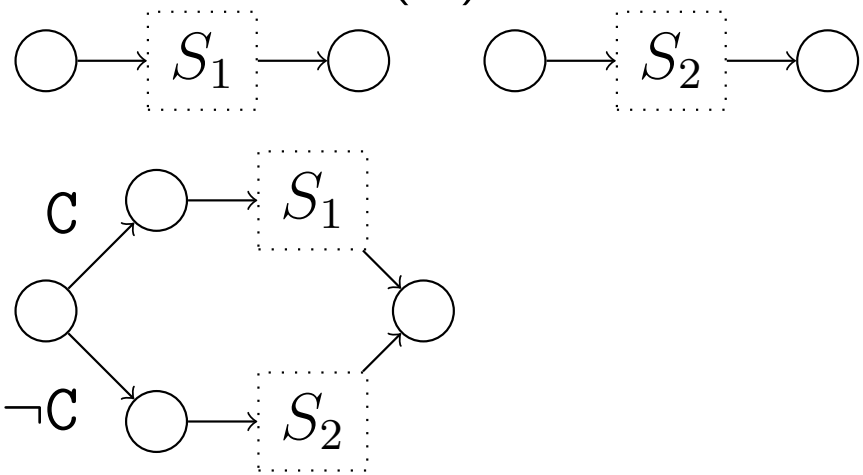
From Source Code to Control-Flow Automaton

Assignment `var=expr;`

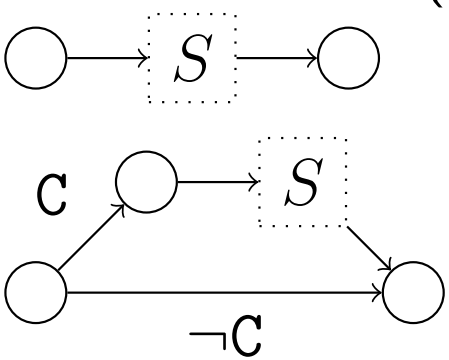
While-Statement `while` (C) S



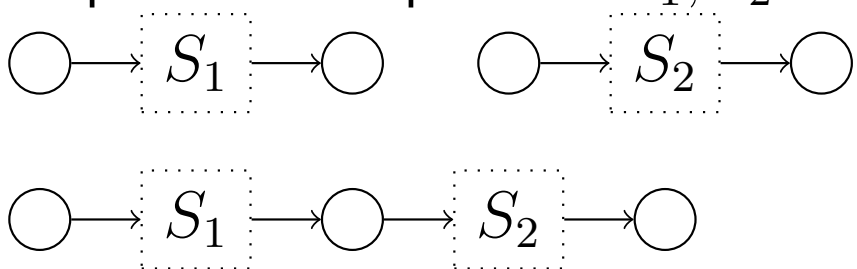
If-Statement `if` (C) S_1 `else` S_2



If-Statement `if` (C) S



Sequential Composition $S_1; S_2$



Semantics

Remember: defines meaning of programs

Different types

- Axiomatic semantics: based on pre- and postconditions, e.g. {true}x=0;{x=0}
- Denotational semantics: function from inputs to outputs
- Operational semantics (✓): defines execution of program

Operational Semantics

Defines program meaning by fixing program execution

- Transitions describe single execution steps
  - Level of assignment or assume
  - Change states
  - Evaluate semantics of expressions in a state
- Execution: sequence of transitions

Concrete States

Pair of program counter and data state ($C = L \times \Sigma$)

- Program counter
  - Where am I?
  - Location in CFA
  - $c(pc)$ refers to program counter of concrete state
- Data state $\sigma : V \to \mathbb{Z}$
  - Fixes variable values
  - $c(d)$ refers to data state of concrete state

Semantics of Arithmetic Expressions

Evaluation function $\mathcal{S}_a : aexpr \times \Sigma \to \mathbb{Z}$

Defined recursively on structure

- $\text{const} \in \mathbb{Z} : \mathcal{S}_a(\text{const}, \sigma) = \text{const}$
- variable `var`: $\mathcal{S}_a(\texttt{var}, \sigma) = \sigma(\texttt{var})$
- unary operation: $\mathcal{S}_a(-t, \sigma) = -\mathcal{S}_a(t, \sigma)$
- binary operation:
  $\mathcal{S}_a(t_1 \ op_a \ t_2, \sigma) = \mathcal{S}_a(t_1, \sigma) \ op_a \ \mathcal{S}_a(t_2, \sigma)$

Semantics of Boolean Expressions

Evaluation function $\mathcal{S}_b : bexpr \times \Sigma \rightarrow \{true, false\}$

Defined recursively on structure

- arithmetic expression:
$$\mathcal{S}_b(t, \sigma) = \begin{cases} true & \text{if } \mathcal{S}_a(t, \sigma) \neq 0 \\ false & \text{else} \end{cases}$$
- comparison: $\mathcal{S}_b(t_1\ op_c\ t_2, \sigma) = \mathcal{S}_a(t_1, \sigma)\ op_c\ \mathcal{S}_a(t_2, \sigma)$
- logic connection: $\mathcal{S}_b(b_1\ op_b\ b_2, \sigma) = \mathcal{S}_b(b_1, \sigma)\ op_b\ \mathcal{S}_b(b_2, \sigma)$

Examples for Expression Evaluation

Consider σ : $\texttt{abs} \mapsto 2; \texttt{i} \mapsto 0; \texttt{x} \mapsto -2$

Derivation of the values of

- $\mathcal{S}_a(-x, \sigma)$
- $\mathcal{S}_a(2 * i, \sigma)$
- $\mathcal{S}_b(x > 0, \sigma)$
- $\mathcal{S}_b(i < abs, \sigma)$

on the board.

State Update

$$\Sigma \times Ops_{\text{assignment}} \to \Sigma$$

$$\sigma[var = aexpr;] = \sigma'$$

$$\text{with } \sigma'(v) = \begin{cases} \sigma(v) & \text{if } v \neq var \\ \mathcal{S}_a(aexpr, \sigma) & \text{else} \end{cases}$$

Examples for State Update

Consider σ : $\texttt{abs} \mapsto 2; \texttt{i} \mapsto 0; \texttt{x} \mapsto -2$

Computation of the state updates

- $\sigma[i = 1;]$
- $\sigma[abs = -x;]$
- $\sigma[i = 2 * i;]$

on the board.

Transitions – Single Execution Steps

Transitions $\mathcal{T} \subseteq C \times G \times C$ with $(c, (l, op, l'), c') \in \mathcal{T}$ if

1. Respects control-flow, i.e.,

$$c(pc) = l \wedge c'(pc) = l'$$

2. Valid data behavior
   - op assignment `var=aexpr;` $\wedge c'(d) = c(d)[var = aexpr;]$
   - op assume `bexpr`
     $\wedge \mathcal{S}_b(\texttt{bexpr}, c(d)) = true \wedge c(d) = c'(d)$

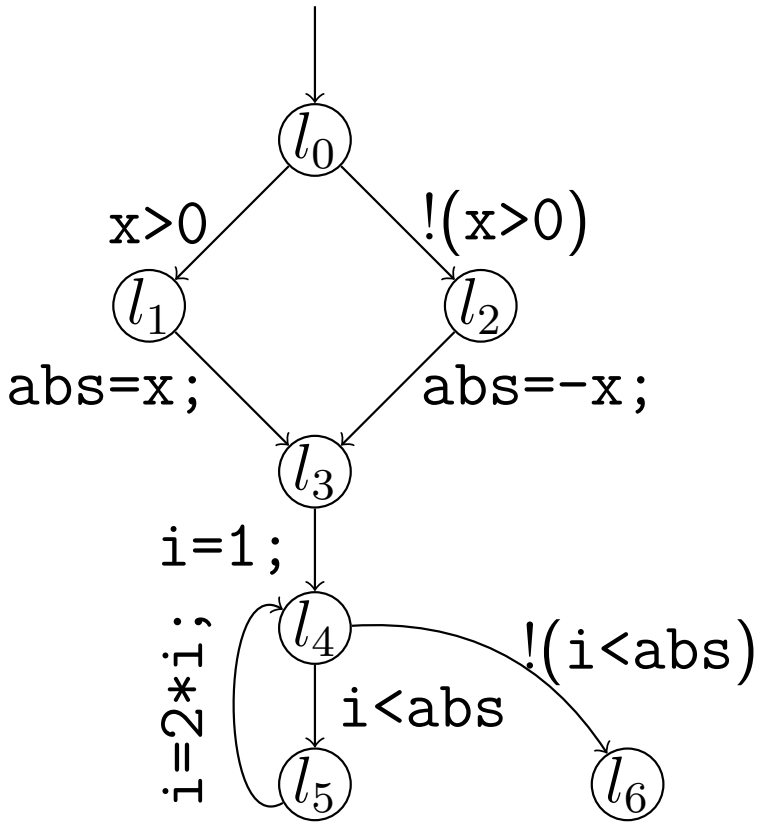
Program Paths

Defined inductively

- every concrete state c with $c(pc) = l_0$
- $c_0 \xrightarrow{g_1} c_1 \cdots \xrightarrow{g_n} c_n$ program path and $(c_n, g_{n+1}, c_{n+1}) \in \mathcal{T}$, then $c_0 \xrightarrow{g_1} c_1 \cdots \xrightarrow{g_n} c_n \xrightarrow{g_{n+1}} c_{n+1}$ program path

Set of all program paths of program $P = (L, G, l_0)$ denoted by $paths(P)$.

Examples for Program Paths



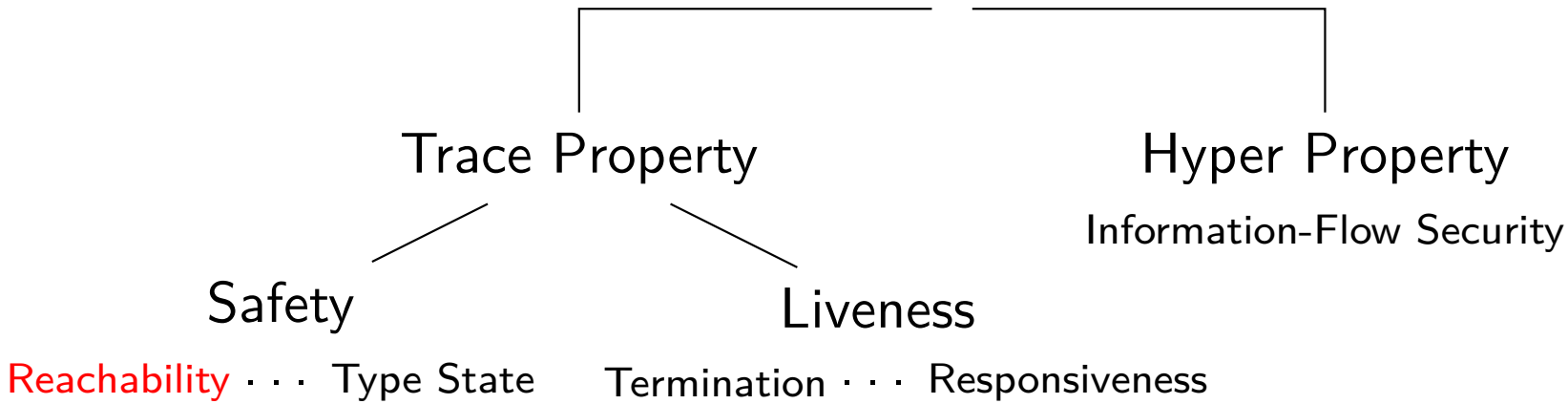
On the board: Shortest and longest program path starting in state (l_0, σ) with $\sigma : \texttt{abs} \mapsto 2; \texttt{i} \mapsto 0; \texttt{x} \mapsto -2$

Reachable States

$$reach(P) := \{c \mid \exists c_0 \xrightarrow{g_1} c_1 \cdots \xrightarrow{g_n} c_n \in paths(P) : c_n = c\}$$

Program Properties and Program Correctness

Program Properties



Reachability Property φ_R

Defines which concrete states $\varphi_R \subseteq C$ must not be reached

In this lecture:

- Certain program locations must not be reached
- Denoted by $\varphi_{L_{\text{sub}}} := \{c \in C \mid c(pc) \in L_{\text{sub}}\}$

Correctness

Definition

Program P is correct wrt. reachability property φ_R if

$$reach(P) \cap \varphi_R = \emptyset.$$

Formalizing Verification Terms

- False alarm: $v(P, \varphi_R) = \text{FALSE} \wedge reach(P) \cap \varphi_R = \emptyset$
- False proof: $v(P, \varphi_R) = \text{TRUE} \wedge reach(P) \cap \varphi_R \neq \emptyset$
- Verifier v is sound and complete if v does not produce false proofs and false alarms, respectively.

Abstract Domains

Problem With Program Semantics

- Infinitely many data states σ
  $\Rightarrow$ infinitely many reachable states

- Cannot analyze program paths individually

How to deal with infinite state space?

Idea: analyze set of program paths together

- Group concrete states $\Rightarrow$ abstract states
- Define (abstract) semantics for abstract states

$\Rightarrow$ Abstract domain

Partial Order (Recap)

Definition

Let E be a set and $\sqsubseteq \subseteq E \times E$ a binary relation on E. The structure $(E, \sqsubseteq)$ is a *partial order* if $\sqsubseteq$ is

- reflexive $\forall e \in E : e \sqsubseteq e$,
- transitive $\forall e_1, e_2, e_3 \in E : (e_1 \sqsubseteq e_2 \wedge e_2 \sqsubseteq e_3) \Rightarrow e_1 \sqsubseteq e_3$,
- antisymmetric
  $\forall e_1, e_2 \in E : (e_1 \sqsubseteq e_2 \wedge e_2 \sqsubseteq e_1) \Rightarrow e_1 = e_2$.

Examples for Partial Orders

- $(\mathbb{Z}, \leq)$
- $(2^Q, \subseteq)$
- $(\Sigma^*, \text{lexicographic order})$
- $(\Sigma^*, \text{suffix})$

Chains

Let $(E, \sqsubseteq)$ be a partial order.

Definition (Chain)

A subset $E_{\text{subset}} \subseteq E$ is a chain if it is totally ordered, i.e. $\forall e, e' \in E_{\text{sub}} : e \sqsubseteq e' \vee e' \sqsubseteq e$.

A chain E_{subset} is finite if the subset E_{subset} is finite.

Ascending Chains

Let $(E, \sqsubseteq)$ be a partial order.

Definition (Ascending Chain)

A sequence $(e_i)_{n \in \mathbb{N}} \in E^\omega$ is an ascending chain if $\forall m, m' \in \mathbb{N} : m \leq m' \Rightarrow e_m \sqsubseteq e_{m'}$.

Definition (Stabilization)

A sequence $(e_i)_{n \in \mathbb{N}} \in E^\omega$ eventually stabilizes if $\exists n_0 \in \mathbb{N} : \forall n \in \mathbb{N} : n \geq n_0 : e_n = e_{n_0}$

Definition (Stabilizing Ascending Chain)

A *stabilizing* ascending chain eventually stabilizes.

Examples for Chains

Consider $(\mathbb{Z}, =)$

- Set {1,2} not a chain
- $(a_1, a_2, \dots)$ with $a_i = 1$ ascending and stabilizing
- Is a stabilizing ascending chain.

Consider $(\mathbb{Z}, \leq)$

- Every subset of $\mathbb{Z}$ is a chain.
- $(a_1, a_2, \dots)$ with $a_i = \begin{cases} 0 & \text{if } i \text{ even} \\ 1 & \text{else} \end{cases}$ not ascending
- $(a_1, a_2, \dots)$ with $a_i = i$ ascending, but not stabilizing
- $(a_1, a_2, \dots)$ with $a_i = min(i, 10)$ ascending and stabilizing
- Is not a stabilizing ascending chain.

Height of Partial Order

Let $(E, \sqsubseteq)$ be a partial order.

- $(E, \sqsubseteq)$ has finite height if all chains are finite.
- $(E, \sqsubseteq)$ has height h if all chains contain at most $h + 1$ elements and one chain contains $h + 1$ elements.

Note: If E is finite than $(E, \sqsubseteq)$ has finite height,
but not vice versa.
For example, $(\mathbb{Z}, =)$

Heights of Example Partial Orders

PO	finite height	height
$(\mathbb{Z}, \leq)$	×	
$(\mathbb{Z}, \geq)$		
$(\mathbb{Z}, =)$	✓	0
$(2^Q, \subseteq)$, Q finite	✓	$\lvert Q \rvert$
$(\Sigma^*, \text{lexicographic order})$		
$(\Sigma^*, \text{suffix})$		

Upper Bound (Join)

Let $(E, \sqsubseteq)$ be a partial order.

Definition (Upper Bound)

An element $e \in E$ is an upper bound of a subset $E_{\text{sub}} \subseteq E$ if

$$\forall e' \in E_{\text{sub}} : e' \sqsubseteq e.$$

Definition (Least Upper Bound (lub))

An element $e \in E$ is a least upper bound $\sqcup$ of a subset $E_{\text{sub}} \subseteq E$ if

- e is an upper bound of E_{sub} and
- for all upper bounds e' of E_{sub} it yields that $e \sqsubseteq e'$.

Lower Bound (Meet)

Let $(E, \sqsubseteq)$ be a partial order.

Definition (Lower Bound)

An element $e \in E$ is an lower bound of a subset $E_{\text{sub}} \subseteq E$ if

$$\forall e' \in E_{\text{sub}} : e \sqsubseteq e'.$$

Definition (Greatest Lower Bound (glb))

An element $e \in E$ is a greatest lower bound $\sqcap$ of a subset $E_{\text{sub}} \subseteq E$ if

- e is a lower bound of E_{sub} and
- for all lower bounds e' of E_{sub} it yields that $e' \sqsubseteq e$.

Computing Upper Bounds

PO	subset	$\sqcup$ (lub)	$\sqcap$ (glb)
$(\mathbb{Z}, \leq)$	$\{1, 4, 7\}$	7	1
$(\mathbb{Z}, \leq)$	$\mathbb{Z}$	$\times$	$\times$
$(\mathbb{N}, \leq)$	$\emptyset$	0	$\times$
$(2^Q, \subseteq)$	2^Q		
$(2^Q, \subseteq)$	$\{\emptyset\}$		
$(2^Q, \subseteq)$	$Y \subseteq 2^Q$		

Facts About Upper and Lower Bounds

1. Least upper bounds and greatest lower bound do not always exist.
   For example,
   - $(\mathbb{Z}, \leq)$
   - $(\mathbb{N}, \leq)$
   - $(\mathbb{N}, \geq)$
2. The least upper bound and the greatest lower bound are unique if they exists.
   (Proof on the board)

Lattice

Definition

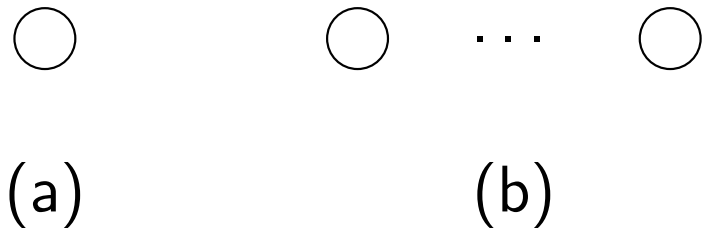
A structure $\mathcal{E} = (E, \sqsubseteq, \sqcup, \sqcap, \top, \bot)$ is a lattice if

- $(E, \sqsubseteq)$ is a partial order
- least upper bound $\sqcup$ and greater lower bound $\sqcap$ exist for all subsets $E_{\text{sub}} \subseteq E$
- $\top = \sqcup E = \sqcap \emptyset$ and $\bot = \sqcap E = \sqcup \emptyset$

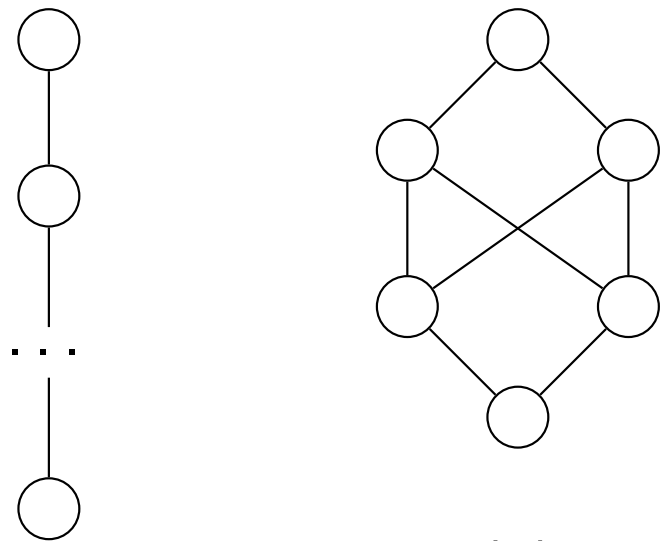
Note:

For any set Q the structure $(2^Q, \subseteq, \cup, \cap, Q, \emptyset)$ is a lattice.

Which Partial Orders Are Lattices?

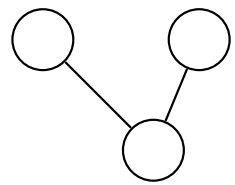


(a) (b)

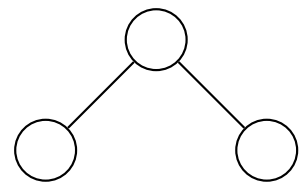


(c)

(d)



(e)



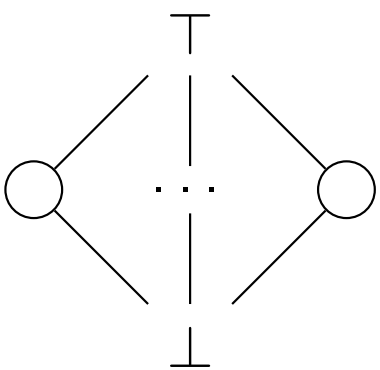
(f)

Flat-Lattice

Definition

A flat lattice of set Q consists of

- Extended set $Q_{\perp}^{\top} = Q \cup \{\top, \perp\}$
- Flat ordering $\sqcup$, i.e. $\forall q \in Q : \perp \sqsubseteq q \sqsubseteq \top$ and $\perp \sqsubseteq \top$
- $\sqcup = \begin{cases} \perp & X = \emptyset \vee X = \{\perp\} \\ q & X = \{q\} \vee X = \{\perp, q\} \\ \top & \text{else} \end{cases}$
- $\sqcap = \begin{cases} \top & X = \emptyset \vee X = \{\top\} \\ q & X = \{q\} \vee X = \{\top, q\} \\ \perp & \text{else} \end{cases}$



Q

Product Lattice

Let $\mathcal{E}_1 = (E_1, \sqsubseteq_1, \sqcup_1, \sqcap_1, \top_1, \bot_1)$ and
$\mathcal{E}_2 = (E_2, \sqsubseteq_2, \sqcup_2, \sqcap_2, \top_2, \bot_2)$ be lattices.

The product lattice $\mathcal{E}_\times = (E_1 \times E_2, \sqsubseteq_\times, \sqcup_\times, \sqcap_\times, \top_\times, \bot_\times)$ with

- $(e_1, e_2) \sqsubseteq_\times (e_1', e_2')$ if $e_1 \sqsubseteq_1 e_1' \wedge e_2 \sqsubseteq_2 e_2'$
- $\sqcup_\times E_{\text{sub}} = (\sqcup_1\{e_1 \mid (e_1, \cdot) \in E_{\text{sub}}\}, \sqcup_2\{e_2 \mid (\cdot, e_2) \in E_{\text{sub}}\})$
- $\sqcap_\times E_{\text{sub}} = (\sqcap_1\{e_1 \mid (e_1, \cdot) \in E_{\text{sub}}\}, \sqcap_2\{e_2 \mid (\cdot, e_2) \in E_{\text{sub}}\})$
- $\top_\times = (\top_1, \top_2)$ and $\bot_\times = (\bot_1, \bot_2)$

is a lattice.

Proof on the board.

Join-Semi-Lattice

Complete lattice always not required
$\Rightarrow$ remove unused elements

Definition

Join-Semi-Lattice A structure $\mathcal{E} = (E, \sqsubseteq, \sqcup, \top)$ is a lattice if

- $(E, \sqsubseteq)$ is a partial order
- least upper bound $\sqcup$ exists for all subsets $E_{\text{sub}} \subseteq E$
- $\top = \sqcup E$

Abstract Domain

Join-semi-lattice on set of abstract states
+ meaning of abstract states

Definition

An abstract domain $D = (C, \mathcal{E}, [\![\cdot]\!])$ consists of

- a set C of concrete states
- a join-semi-lattice $\mathcal{E} = (E, \sqsubseteq, \sqcup, \top)$
- a concretization function $[\![\cdot]\!] : E \to 2^C$
  (assigns meaning of abstract states)
  - $[\![\top]\!] = C$
  - $\forall E_{\text{sub}} \subseteq E : \bigcup_{e \in E_{\text{sub}}} [\![e]\!] \subseteq [\![\sqcup E_{\text{sub}}]\!]$
    (join operator overapproximates)

Abstraction

$$\alpha : 2^C \rightarrow E$$

Here:

- Not defined separately
- Returns smallest abstract state that covers set of concrete states

Galois Connection

Abstraction and concretization function fulfill the following connection

1. $\forall C_{\text{sub}} \subseteq C : C_{\text{sub}} \subseteq [\![\alpha(C_{\text{sub}})]\!]$
   (abstraction safe approximation,
   but may loose information/precision)
2. $\forall e \in E : \alpha([\![e]\!]) \sqsubseteq e$
   (no loss in safety)

Abstract Semantics

Abstract interpretation of program, i.e., evaluation on abstract states

Transfer relation $\rightsquigarrow \subseteq E \times G \times E$

- $\forall e \in E, g \in G$:
  $\bigcup_{c \in [\![e]\!]} \{c' \mid (c, g, c') \in \mathcal{T}\} \subseteq \bigcup_{(e,g,e') \in \rightsquigarrow} [\![e']\!]$
  (safe over-approximation)
- Depends on abstract domain
- In this lecture: restricted to functions

Properties of Transfer Functions

- Monotony

$$\forall e, e' \in E, g \in G : e \sqsubseteq e' \Rightarrow \rightsquigarrow (e, g) \sqsubseteq \rightsquigarrow (e', g)$$

- Distributivity

$$\forall e, e' \in E, g \in G : \rightsquigarrow (e, g) \sqcup \rightsquigarrow (e', g) = \rightsquigarrow (e \sqcup e', g)$$

Elements of Abstraction (Recap.)

1. Abstract domain
   - Join-semi lattice $\mathcal{E}$ on set of abstract states E
   - Meaning of abstract states $[\![]\!]$
2. Abstract semantics $\rightsquigarrow$

Properties of Abstraction (Recap.)

- Join operator overapproximates

$$\forall E_{\text{sub}} \subseteq E : \bigcup_{e \in E_{\text{sub}}} [\![e]\!] \subseteq [\![\sqcup E_{\text{sub}}]\!]$$

- Monotony of transfer relation

$$\forall e, e' \in E, g \in G : e \sqsubseteq e' \Rightarrow \rightsquigarrow (e, g) \sqsubseteq \rightsquigarrow (e', g)$$

- Distributivity of transfer relation

$$\forall e, e' \in E, g \in G : \rightsquigarrow (e, g) \sqcup \rightsquigarrow (e', g) = \rightsquigarrow (e \sqcup e', g)$$

Location Abstraction $\mathbb{L}$

Tracks control-flow of program

- Uses flat lattice of set L of location states
- $[\![\ell]\!] := \begin{cases} C & \text{if } \ell = \top \\ \emptyset & \text{if } \ell = \bot \\ \{c \in C \mid c(pc) = \ell\} & \text{else} \end{cases}$
  (guarantees that join overapproximates)
- $(\ell, (l, op, l'), \ell') \in \rightsquigarrow_{\mathbb{L}}$ if $(\ell = l \vee \ell = \top)$ and $\ell' = l'$

Properties of Location Abstraction

Transfer relation $\rightsquigarrow_{\mathbb{L}}$

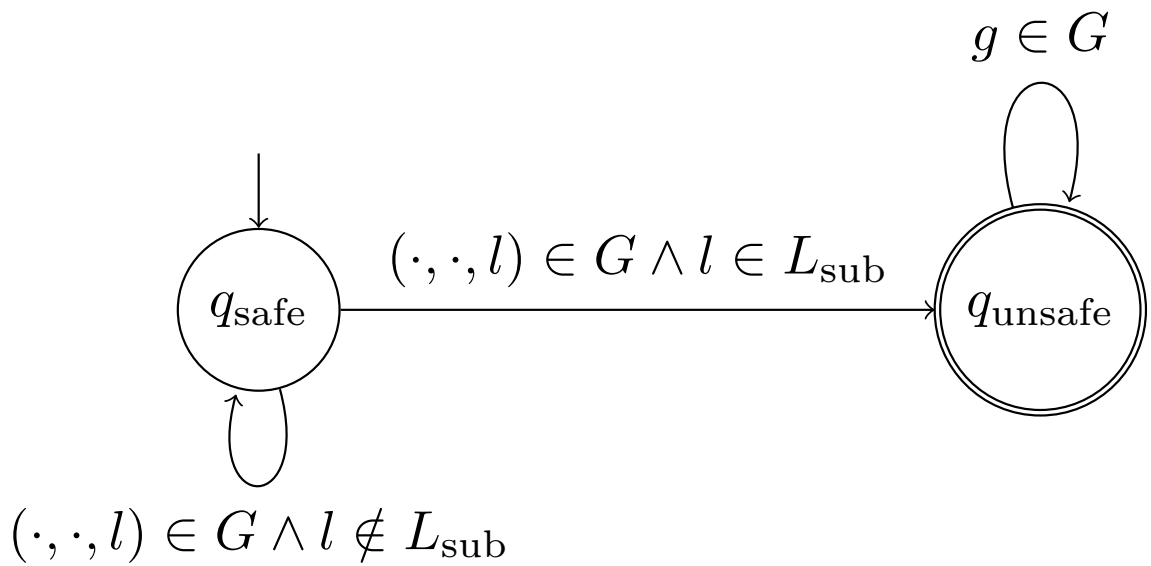
- overapproximates, i.e.,

$$\forall e \in E_{\mathbb{L}}, g \in G : \bigcup_{c \in [\![e]\!]} \{c' \mid (c, g, c') \in \mathcal{T}\} \subseteq \bigcup_{(e,g,e') \in \rightsquigarrow_{\mathbb{L}}} [\![e']\!]$$

- monotone
- distributive

Property Encoding

Automaton observing violation of reachability property $\varphi_{L_{\text{sub}}}$



Property Abstraction $\mathbb{R}$

Represent automaton encoding of property $\varphi_{L_{\text{sub}}}$ as abstraction

- Uses join-semilattice on set $\{q_{\text{safe}}, q_{\text{unsafe}}\}$ with $q_{\text{safe}} \sqsubseteq q_{\text{unsafe}}$
- $[\![q]\!] := \begin{cases} C & \text{if } q = q_{\text{unsafe}} \\ \{c \in C \mid c(pc) \notin L_{\text{sub}}\} & \text{else} \end{cases}$
- $(q, (l, op, l'), q') \in \rightsquigarrow_{\mathbb{R}}$ if $q' = q_{\text{unsafe}} \wedge l' \in L_{\text{sub}}$ or $q = q \wedge l' \notin L_{\text{sub}}$

Properties of Property Abstraction

Transfer relation $\rightsquigarrow_{\mathbb{R}}$

- overapproximates
- monotone
- distributive

Variable Separate Domain

Assigns to each variable an abstract value from base domain $B = (E_B, \sqsubseteq_B, \sqcup_B, \top_B)$

- $E = \{f : Var \to B\}$
- $f \sqsubseteq f'$ if $\forall v \in Var : f(v) \sqsubseteq_B f'(v)$
- $\sqcup F = f''$ with $\forall v \in Var : f''(v) = \sqcup_{f \in F} f(v)$
- $[\![f]\!] = \{c \mid \forall v \in Var : c(d)(v) \in [\![f(v)]\!]_B\}$

Value Abstraction $\mathbb{V}$

Uses variable separate domain

- Base domain flat lattice of $\mathbb{Z}$
- Abstract value $\top$ any value
- Transfer relation
  - Assignment $(f, (l, v := expr;, l'), f') \in \rightsquigarrow_{\mathbb{V}}$ if
    $\forall w \in Var : v \neq w \Rightarrow f(w) = f'(w)$ and $f'(v) =$
    $$\begin{cases} \top & \text{if } \exists w \in var(expr) : f(w) = \top \\ \bot & \text{if } \exists w \in var(expr) : f(w) = \bot \\ \mathcal{S}_a(expr, f) & \text{else} \end{cases}$$
  - Assume $(f, (l, expr, l'), f) \in \rightsquigarrow_{\mathbb{V}}$ if
    $\exists w \in var(expr) : f(w) = \top$ or
    $\forall w \in var(expr) : f(w) \in \mathbb{Z} \wedge \mathcal{S}_b(expr, f)$

Properties of Value Abstraction $\mathbb{V}$

Transfer relation

- overapproximates
- monotone
- not distributive, e.g.,

$f : x \mapsto 3; y \mapsto 2 \qquad f' : x \mapsto 2; y \mapsto 3$
$\rightsquigarrow (f, x = x + y;) \sqcup \rightsquigarrow (f', x = x + y;) : x \mapsto 5; y \mapsto \top,$
but $\rightsquigarrow (f \sqcup f', x = x + y;) : x \mapsto \top; y \mapsto \top$

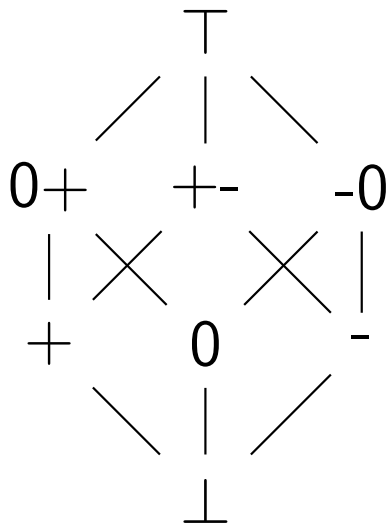
Example Abstract Transitions

On the board:

- $\rightsquigarrow (i \mapsto \top; x \mapsto 3, (l, i = 1; , l'))$
- $\rightsquigarrow (i \mapsto \top; x \mapsto \top, (l, i = i * 2; , l'))$
- $\rightsquigarrow (i \mapsto \top; x \mapsto 5, (l, i = i * 2; , l'))$
- $\rightsquigarrow (i \mapsto 0; x \mapsto \top, (l, i \&\& x > 0, l'))$
- $\rightsquigarrow (i \mapsto \top; x \mapsto 10, (l, x > 10, l'))$

Sign Abstraction

Variable separate domain using base domain



$$[\![\top]\!] = \mathbb{Z} \quad [\![+]\!] = \mathbb{N}^+ \quad [\![-]\!] = \mathbb{Z} \setminus \mathbb{N}_0^+ \quad [\![0]\!] = \{0\}$$
$$[\![+-]\!] = \mathbb{Z} \setminus \{0\} \quad [\![0+]\!] = \mathbb{N}_0^+ \quad [\![-0]\!] = \mathbb{Z} \setminus \mathbb{N}^+ \quad [\![\bot]\!] = \emptyset$$

Transfer Relation of Sign Abstraction

Suggestion 1:

- $\rightsquigarrow (f, g) = f'$ with $\forall v \in Var : f'(v) = \top$
- sound, but not useful

Transfer Relation of Sign Abstraction

Suggestion 2:

- Assume: $\rightsquigarrow (f, expr) = f$
- Assignment: $\rightsquigarrow (f, expr) = f'$

$$\text{v=const;} \quad f'(v) = \begin{cases} + & const \in \mathbb{N}^+ \\ 0 & const = 0 \\ - & \text{else} \end{cases}$$

$$\text{v=w;} \quad f'(v) = f(v)$$

$$\text{v=expr;} \quad f'(v) = \top$$

and $\forall u \in Var : u \neq v \Rightarrow f'(u) = f(u)$

sound, but could be more precise

Transfer Relation of Sign Abstraction (Incomplete)

More precise for special boolean expression like
var>0, var==0, var<0, var>=0, var<=0

- can be decided
- used to restrict successor of assume expressions

Abstract evaluation of arithmetic expressions, e.g.

- e + e = e, for any abstract value e except +-
- e + 0 = e
- e - 0 = e
- e*0 = 0
- ...

Interval Abstraction $\mathbb{I}$

Variable separate domain based on interval domain

- $E = \mathbb{Z}^2 \cup \{\top, \bot\}$
- $\bot \sqsubseteq e$, $e \sqsubseteq \top$ and $[a, b] \sqsubseteq [c, d]$ if $c \leq a \wedge b \leq d$
- $$\sqcup E_{\text{sub}} = \begin{cases} \top & \text{if } \top \in E_{\text{sub}} \\ \bot & \text{if } E_{\text{sub}} \subseteq \{\bot\} \\ [\min_{[a,b] \in E_{\text{sub}}} a, \max_{[a,b] \in E_{\text{sub}}} b] & \text{else} \end{cases}$$
- $[\![[a, b]]\!] = \{x \in \mathbb{Z} \mid a \leq x \leq b\} \quad [\![\top]\!] = \mathbb{Z} \quad [\![\bot]\!] = \emptyset$

Violates ascending chain condition.

Transfer Relation of Interval Abstraction

Relies on abstract evaluation of expressions in state f

Arithmetic expressions

- const: [const,const]
- var: f(var)
- -[a,b]=[-b,-a]
- [a,b] op_a [c,d] = $[\min(a\ op_a\ c, b\ op_a\ d), \max(a\ op_a\ c, b\ op_a\ d)]$
- special treatment of values $\bot, \top$

Transfer Relation of Interval Abstraction

Relies on abstract evaluation of expressions in state f

Boolean expression

- [a,b]$= \begin{cases} \{true\} & a > 0 \vee b < 0 \\ \{false\} & a = b = 0 \\ \{true, false\} & \text{else} \end{cases}$
- [a,b]<[c,d]$= \begin{cases} \{true\} & b < c \\ \{false\} & a \geq d \\ \{true, false\} & \text{else} \end{cases}$
- other comparison operators similar
- ...

Define transfer relation analogous to transition

Cartesian Predicate Abstraction

Represent states by first order logic formulae

- Restricted to a set of predicates Pred
  (subset of boolean expressions without boolean connectors)
- Conjunction of predicates

Cartesian Predicate Abstraction

- Power set lattice on predicates $(2^{\text{Pred}}, \supseteq, \cap, \cup, \emptyset, Pred)$

- $[\![\top]\!] = [\![\emptyset]\!] = C$
  for $p \neq \bot$:
  $[\![p]\!] = \{c \in C \mid \forall pred \in p : \mathcal{S}_b(pred, c(d)) = true\}$
  (guarantees that join overapproximates)

- Transfer relation
  - Assignment
    $(p, v = expr, p')$ with
    $p' = \left\{ t \in Pred \;\middle|\; \left(\exists v' : \bigwedge_{t' \in p} t'[v \to v'] \wedge v = expr[v \to v'] \right) \Rightarrow t \right\}$
  - Assume
    $(p, expr, p')$ if $\bigwedge_{t \in p} t \wedge expr$ is satisfiable and
    $p' = \{t \in Pred \mid (\bigwedge_{t' \in p} t' \wedge expr) \Rightarrow t\}$

Properties of Cartesian Predicate Abstraction

Transfer relation

- overapproximates
- monotone
- not distributive
  (e.g., use value abstraction example and value assignments as predicates)

Example Abstract Transitions

Consider set of predicates {i>0, x=10}

On the board:

- $\rightsquigarrow (\{x = 10\}, (l, i = 1; , l'))$
- $\rightsquigarrow (\{i > 0\}, (l, i = i * 2; , l'))$
- $\rightsquigarrow (\{i > 0\}, (l, i < abs, l'))$
- $\rightsquigarrow (\{x = 10, i > 0\}, (l, x > 10, l'))$

Composite Abstraction

Combines two abstractions

- Product (join-semi) lattice $E_1 \times E_2$
- $[\![(e_1, e_2)]\!] = [\![e_1]\!]_1 \cap [\![e_2]\!]_2$
- Product transfer relation
  $((e_1, e_2), g, (e'_1, e'_2)) \in \rightsquigarrow$
  if $(e_1, g, e'_1) \in \rightsquigarrow_1$ and $(e_2, g, e'_2) \in \rightsquigarrow_2$
- More precise transfer relations possible

Properties of Composite Abstraction

Properties inherited from components

Transfer relation

- overapproximates
- monotone
- distributive

if respective property is fulfilled by both components.

Proof on the board

Two Prominent Combination

- Value analysis $\mathbb{L} \times \mathbb{V} \times \mathbb{R}$
- Predicate analysis $\mathbb{L} \times \mathbb{P} \times \mathbb{R}$